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Uniform Asymptotic Stability of Functional Differential Equations with Infinite Delay

Abstract

In this paper criteria on uniform asymptotic stability is established for impulsive functional differential equations with infinite delay. It is shown that certain impulsive perturbations may make unstable systems uniformly stable, even uniformly asymptotically stable.

Keywords: Stability, Impulsive Delay Differential Equations, Razumikhin Technique, Lyapunov Function

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Introduction

Impulsive differential equations arise naturally from a wide variety of applications such as aircraft control, inspection process in operations research and threshold theory in biology. Significant progress has been made in the theory of differential equations in recent years. But still there are number of difficulties one may face in developing the corresponding theory of impulsive delay differential equations. For example in the classical theory of delay differential equations, the fact that the continuity of a function $x(t)$ in R^n implies the continuity of the functional x_t in C^n, plays a key role in establishing the existence of solution of delay differential equations [1]. However if the function $x(t)$ is piecewise continuous, then the functional x_t need not be piecewise continuous. In fact, it can be discontinuous everywhere. Existence and uniqueness results for impulsive delay differential equations have been presented in [2]. In [6,7], by using Lyapunov functions and Razumikhin techniques, some Razumikhin type theorems on stability are obtained for a class of impulsive functional differential equations with finite delay. However as pointed out in [8-10] even though for functional differential equationswithout impulses, stability results established for equations with finite delay are not obviously true in general for infinite delays. The common and main difficulty is that the interval $(-\infty, t_0]$ is not compact and the images of a solution map of closed and bounded sets in $((-\infty, t_0]\ ,R^n)$ space may not be compact. Same situation arises in $((-\infty, t_0]\ ,R^n)$ space for impulsive differential equation with infinite delay. Recall that the stability theory of impulsive differential equations with infinite delays had received much attention in the literature [11-16].Here we extend the result develop in [3] to study infinite delay differential equations.

Aim of the Study

The purpose of present paper is to establish some criteria on uniform asymptotic stability for impulsive differential equations with infinite delay using Lyapunov functions and Razumikhin techniques.

Review of Literature

In past years there have been intensive studies on the stability of Impulsive Differential equations. In 1991, M. ramamohana Rao, investigates sufficient condition for uniform stability and uniform asymptotic stability of impulsive integro differential equations by constructing suitable piecewise continuous Lyapunov-like functionals without the decresent property.M. U. Akhmet,investigate the sufficient criteria for stability, asymptotic stability and instability for non-trivial solutions of the impulsive systems by Lyapunov's second method.Jianhua Shen and Jianli Li, investigates the sufficient criteria on asymptotic stability for system of volterra functional differential equations with nonlinear impulsive perturbations using Lyapunov like functions with Razumikhin technique or Lyapunov like functional.

Preliminaries

Let R^n be a space of n dimensional column vectors $x = col(x_1, \dots\dots\dots, x_n)$ with some norm $\|.\|$. Let $J \subset R$ be any interval of the form $[a,b)$ where $0 \le a < b \le \infty$ and $D \subset R^n$ be an open set. Consider the system

$$x'(t) = F(t, x(.)), \qquad t > t^* (2.1)$$
$$\Delta x(t_k) = I(t_k, x(t_k^-)), \quad k = 1,2, \dots.. \qquad (2.2)$$

Where $x'(t)$ denote the right hand derivative of $x(t)$, $t^* < t_k < t_{k+1}$ with $t_k \to \infty$ as $k \to \infty$, $I: [t^*, \infty) \times R^n \to R^n$ and $x(t_k^-) = \lim_{t \to t_k - 0} x(t)$

Define $PC(J, R^n) = [x: J \to R^n | x$ is continuous everywhere except at the points $t = t_k \in J$ and $x(t_k^-)$ and $x(t_k^+) = \lim_{t \to t_k + 0} x(t)$ exist with $x(t_k^+) = x(t_k)]$

For any $t \ge \sigma, PC([\alpha, t], R^n)$ will be written as $PC(t)$.

Define $PCB(t) = [x \in PC(t) | x$ is bounded]. For any $\emptyset \in PCB(t)$, the norm of $\emptyset$ is defined by $\|\emptyset\| = \|\emptyset\|^{[\alpha,t]} = \sup_{\alpha \le s \le t} |\emptyset(s)|$. For any $\sigma \ge t^*$ and $\emptyset \in PCB(\sigma)$ with equations (2.1) and (2.2), one associates an initial condition of the form

$$x_\sigma(t) = \emptyset(t); \ \alpha \le t \le \sigma \qquad (2.3)$$

Definition 2.1

The trivial solution of (2.1) is said to be

i. *stable* if for every $\epsilon > 0$ and $\sigma \in R^+$, there exists some $\delta = \delta(\sigma, \epsilon) > 0$ such that if $\varphi \in PC([\alpha, t], R^n)$ with $\|\varphi\|_r \le \delta$ and $x = x(\sigma, \emptyset)$ is any solution of (2.1) and (2.2) then $x(t, \sigma, \varphi)$ is defined and $\|x(t, \sigma, \varphi)\| \le \epsilon$ for all $t \ge \sigma$

ii. *Uniformly stable* if δ in (i) is independent of σ

iii. *Asymptotically stable* if (i) holds and for any $\sigma \in R$, there exists some $\eta = \eta(\sigma) > 0$ such that if $\varphi \in PC[(\alpha, t), R^n]$ with $\|\varphi\|_r \le \eta$, then $\lim_{t \to \infty} x(t, \sigma, \varphi) = 0$

iv. *Uniformly asymptotically stable* if (ii) holds and there exists some $\eta > 0$ such that for every $\gamma > 0$, there exist some $T = T(\eta, \gamma) > 0$ such that if $\varphi \in PC[(\alpha, t), R^n]$ with $\|\varphi\|_r \le \eta$ then $\|x(t, \sigma, \varphi)\| \le \gamma$ for all $t \ge \sigma + T$

Definition 2.2

A function $x(t)$ is called a solution corresponding to σ of the initial value problem (2.1) – (2.3) if $x: [\alpha, \beta) \to R^n$ (for some $t^* < \beta < \infty$) is continuous for $t \in [\alpha, \beta) \setminus \{t_k, k = 1,2, \dots..\}$, $x(t_k^+)$ and (x_k^-) exist and $x(t_k^+) = x(t_k)$, and satisfies (2.1) – (2.3)

Under the following hypothesis (H_1) – (H_4), the initial value problem (2.1) - (2.3) exists with unique solution which will be written in the form $x(t, \sigma, \varphi)$ (see [5])

(H_1) F is continuous on $[t_{k-1}, t_k) \times PC(t)$ for $k = 1,2, \dots..$ where $t_0 = t^*$. For all $\emptyset \in PC(t)$ and $k = 1,2, \dots..$, the limit $\lim_{(t,\emptyset) \to (t_k^-, \varphi)} F(t, \emptyset) = F(t_k^-, \varphi)$ exists.

(H_2) F is locally Lipchitz in $\emptyset$ in each compact set in $PCB(t)$. More precisely, for any $\gamma \in [\alpha, \beta)$ and every compact set $G \subset PCB(t)$, there exist a constant $L = L(\gamma, G)$ such that $|F(t, \varphi(.)) - F(t, \psi(.))| \le L\|\varphi - \psi\|^{[\alpha,t]}$ whenever $t \in [\alpha, \gamma]$ and $\varphi, \psi \in G$

(H_3) For each $k = 1,2, \dots\dots, I(t, x) \in C([t^*, \infty) \times R^n, R^n)$ and for any $\rho > 0$, there exist a $\rho_1 > 0$ $(0 < \rho_1 < \rho)$ such that $x \in S(\rho_1)$ implies that $x + I(t_k, x) \in S(\rho)$ for $k \in Z^+$

(H_4) For any $x(t) \in PC([\alpha, \infty), R^n), F(t, x(.)) \in PC([t^*, \infty), R^n)$

For any $t \ge t^*, h > 0$ let $PCB_h(t) = P\{\emptyset \in PCB(t): \|\emptyset\| < h\}$

We assume that $F(t, 0) \equiv 0, I(t_k, 0) \equiv 0$ so that $x(t) \equiv 0$ is a solution of (2.1) and (2.2), which we call the zero solution. Also, we will only consider the solutions $x(t, \sigma, \varphi)$ of equations (2.1) and (2.2).

Let us define following class of functions for later use:

$K_1 = \{g \in C(R^+, R^+) | g(0) = 0$ and $g(s) > 0$ for $s > 0\}$;

$K_2 = \{g \in C(R^+, R^+) | g(0) = 0$ and $g(s) > 0$ for $s > 0$ and g is non decreasing in $s\}$.

Main Results

Theorem 3.1

Assume that there exist functions $a, b, c \in K_1, p, q \in PC(R^+, R^+), g \in K_2$ and $V: [t^*, \infty) \times S(\rho) \to R^+$, where V is continuous on $[t_{k-1}, t_k) \times S(\rho_1)$ for $k = 1,2, \dots..$, such that $q(s)$ is non increasing with $q(s) > 0$ for $s > 0$. Assume that the following conditions hold:

i. $b(\|x\|) \le V(t, x(t)) \le a(\|x\|)$ *for all* $(t, x) \in [t^*, \infty) \times S(\rho)$

ii. $V'(t, x(t)) \le p(t) c(V(t, x(t))$ *for any solution* $x(t) = x(t, \sigma, \varphi)$ *of (2.1) and (2.2) whenever* $V(t, x(t)) > g(V(s, x(s))$ *for* $max\{\alpha, t -$ *qVt,xt≤s≤t;*

iii. $V(t_k, x + I(t_k, x)) \le g(V(t_k^-, x))$ for each $k \in Z^+$ and all $x \in S(\rho_1)$

iv. $\tau = \sup_{k \in Z^+} \{t_k - t_{k-1}\} < \infty$, $M_1 = \sup_{k \in Z^+} \int_{t_k}^{t_{k+1}} p(s) ds < \infty$ and $M_2 = \inf_{u > 0} \int_{g(u)}^{u} \frac{ds}{c(s)} > M_1$

Then the zero solution of (2.1) and (2.2) is uniformly asymptotically stable.

Proof

Condition (i) implies $b(s) \le a(s)$ for all $s \in [0, \rho]$. So let $\hat{a}$ and $\hat{b}$ be continuous, strictly increasing functions satisfying $\hat{b}(s) \le b(s) \le a(s) \le \hat{a}(s)$, for all $s \in [0, \rho]$. Then

$$\hat{b}(\|x\|) \le V(t, x) \le \hat{a}(\|x\|) (3.1)$$

for all $(t, x) \in [t^*, \infty) \times S(\rho)$.

From the definition of (M_2), we see that $0 < g(u) < u$ for all $u > 0$.

We first show the uniform stability.

Let $\epsilon > 0$ be given and assume without loss of generality that $\epsilon \le \rho_1$. Choose a positive number $\delta = \delta(\epsilon) > 0$ so that $\delta < \hat{a}^{-1}(g(\hat{b}(\epsilon)))$ and note that

$0 < \delta < \epsilon$.Let$\sigma \geq t^*$ and $\varphi \in PCB_\delta(\sigma)$ and $x(t) = x(t,\sigma,\varphi)$ be the solution of (2.1) and (2.2).

Let $(\sigma,\varphi) \in R_+ \times PC([t^*,0],D)$where$\|\varphi\|_r \leq \delta$.

Set $V(t) = V(t,x(t))$ and let $\sigma \in [t_{l-1},t_l)$ for some $l \in Z^+$ where $t_0 = t^*$. We will prove that $\|x(t)\| \leq \epsilon$ for $\alpha \leq t \leq \sigma$. Suppose for the sake of contradiction that $\|x(t)\| > \epsilon$ for some $t \in [\sigma,\infty)$.Then let $\hat{t} = \inf\{t \geq \sigma | \|x(t)\| > \epsilon\}$. Note that $\|x(t)\| < \epsilon$, we see that $\hat{t} > \sigma$, $\|x(t)\| \leq \epsilon \leq \rho_1$ for $t \in [\sigma,\hat{t})$ and either $\|x(t)\| = \epsilon$ or $\|x(t)\| > \epsilon$ and $\hat{t} = t_k$for some k.In the latter case, $\|x(t)\| \leq \rho$ since $\|x(\hat{t}^-)\| \leq \epsilon \leq \rho_1$ and by our assumption on the functional I.Thus, in either case $V(t)$ is defined for $[\alpha,\hat{t}]$.

For $t \in [\alpha,\hat{t}]$, define

$$m(t) = V\big(t,x(t)\big) \quad (3.2)$$

By the piecewise continuity assumption on V, it follows that $m \in PC([[\alpha,\hat{t}],R_+)$ and $m(t)$ is continuous at each $t \neq t_k$ in $(\sigma,\hat{t}]$.By (3.1), we have

$$\hat{b}(\|x(t)\|) \leq m(t) \leq \hat{a}(\|x(t)\|)(3.3)$$

for $t \in [\alpha,\hat{t}]$.

Thus $m(t) \leq \hat{a}(\|\varphi\|_r) \leq \hat{a}(\delta) < g(\hat{b}(\epsilon))fort \in [\alpha,\sigma]$.

Let $\tilde{t} = \inf\{t \in [\sigma,\hat{t}] | m(t) \geq \hat{b}(\epsilon)\}$. Since $m(\sigma) < g(\hat{b}(\epsilon))$<$\hat{b}(\epsilon)$ and $m(\hat{t}) \geq \hat{b}(\epsilon)$,then$\tilde{t} \in (\sigma,\hat{t})$. Moreover$m(t) < \hat{b}(\epsilon)fort \in (\sigma,\tilde{t})$. We claim that $m(\tilde{t}) = \hat{b}(\epsilon)$ and that $\tilde{t} \neq t_k$ for some k. Clearly we must have $m(\tilde{t}) \geq \hat{b}(\epsilon) > 0$. If $\tilde{t} = t_k$, for some k, then$0 < \hat{b}(\epsilon) \leq m(\tilde{t}) \leq g\big(m(\tilde{t}^-)\big) < m(\tilde{t}^-) \leq \hat{b}(\epsilon)$. By assumption (iii), which is impossible.

Thus $\tilde{t} \neq t_k$ for any k, and that in turn implies that $m(\tilde{t}) = \hat{b}(\epsilon)$, since $m(t)$ is continuous at t.

Next we consider two possible cases:

Case I:$t_{l-1} \leq \sigma < \tilde{t} < t_l$. Let $\bar{t} = \sup\{t \in [\sigma,\tilde{t}] | m(t) \leq g(\hat{b}(\epsilon))\}$.

Since $m(\sigma) < g\left(\hat{b}(\epsilon)\right)$, $m(\tilde{t}) = \hat{b}(\epsilon) \geq g\left(\hat{b}(\epsilon)\right)$ and $m(t)$ is continuous on $[\sigma,\tilde{t}]$, then $\bar{t} \in [\sigma,\tilde{t}], m(\bar{t}) = g(\hat{b}(\epsilon))$ and $m(t) \geq g\left(\hat{b}(\epsilon)\right)fort \in [\bar{t},\tilde{t}\,]$. Hence for $t \in [\bar{t},\tilde{t}\,]$ and $\alpha \leq s \leq t$, we have $g\big(m(s)\big) \leq g\left(\hat{b}(\epsilon)\right) \leq m(t)$. In view of condition (ii), we have for all $t \in [\bar{t},\tilde{t}]$, $m'(t) \leq p(t)c(V(t))$ and so

$$\int_{m(\bar{t})}^{m(\tilde{t})} \frac{ds}{c(s)} \leq \int_{\bar{t}}^{\tilde{t}} p(s)ds \leq \int_{t_{l-1}}^{t_l} p(s)ds \leq M_1 (3.4)$$

However, we have

$$\int_{m(\bar{t})}^{m(\tilde{t})} \frac{ds}{c(s)} = \int_{g\left(\hat{b}(\epsilon)\right)}^{\hat{b}(\epsilon)} \frac{ds}{c(s)} \geq M_2 \quad (3.5)$$

This contradicts the assumption $M_1 < M_2$

Case II:$t_k < \tilde{t} < t_{k+1}$ for some k. Then $m(t_k) \leq g(m(t_k{}^-)) \leq g(\hat{b}(\epsilon))$ by condition (iii). Similar to before, define $\bar{t} = \sup\{t \in [t_k,\tilde{t}] | m(t) \leq g\left(\hat{b}(\epsilon)\right)\}$ then $\bar{t} \in [t_k,\tilde{t})$, $m(\bar{t}) = g\left(\hat{b}(\epsilon)\right)$ and $m(t) \leq g\left(\hat{b}(\epsilon)\right)$ for $t \in [\bar{t},\tilde{t}]$.Applying exactly the same argument as before yields a contradiction.

So, in either case, we obtain a contradiction, which proves that the zero solution of (2.1) and (2.2) is uniformly stable.

Next we show that it is uniformly asymptotically stable.

Since the zero solution of (2.1) and (2.2) is uniformly stable, then there exist some $\delta > 0$ such that if $\varphi \in PCB_\delta(\sigma)$then$\|x(t)\| \leq \rho_1, V(t) \leq \hat{b}(\epsilon),\ t \geq \alpha$.

Now define

$$M = \sup\left\{\frac{1}{c(s)} \mid g\left(\hat{b}(\epsilon)\right) \leq s \leq \hat{a}(\rho_1)\right\} \quad (3.6)$$

and note that$0 < M < \infty$ and so

$$M_2 \leq \int_{g(u)}^{u} \frac{ds}{c(s)} \leq M[u - g(u)] \quad (3.7)$$

from which we obtain $g(u) \leq u - \frac{M_2}{M} \leq u - d$ where $d = d(\epsilon)$ is chosen so that $d < (M_2 - M_1)/M$

Let $w = q(g\left(\hat{b}(\epsilon)\right))$ and N be the positive integer such that $\hat{a}(\rho_1) \leq \hat{b}(\epsilon) + Nd$ and define $T = T(\epsilon) = \tau + (w+\tau)(N-1)$.

Now we will prove that $\|x(t)\| \leq \epsilon$for$t \geq \sigma + T$. Let

$$m(t) = V\big(t,x(t)\big)(3.8)$$

for $t \geq \alpha$, then $m(t) \leq \hat{b}(\rho_1)$ for $t \geq \alpha$. Define the indicies$l_i$for$i = 1,2,\dots\dots,N$ as follows:

Let $l_i = l$ and l_i be chosen so that $t_{l_i-1} < t_{l_{i-1}} + w \leq t_{l_i}$ for $i = 1,2,\dots\dots,N$. Then $t_{l_1} = t_l \leq \sigma + \tau$ and $t_{l_i} \leq t_{l_i-1} + \tau \leq t_{l_{i-1}} + w + \tau$for$i = 1,2,\dots\dots,N$. Let$0 < A - id \leq \hat{a}(\rho_1)$.

We will prove that

$$m(t) \leq A - id \quad (3.9)_i$$

for$t > t_{l_i}$, $i = 1,2,\dots\dots,N$.

To prove$(3.9)_i$, suppose for the sake of contradiction that there exist some $t \geq t_{l_1} = t_l$ for which $m(t) > A - id$. Then let $\tilde{t} = \inf\{t \geq t_l | m(t) > A - id\}$. Thus $\tilde{t} \in [t_k,t_{k+1})$ for some $k \geq l$. Since $m(t_k) \leq g\big(m(t_k{}^-)\big) \leq g(A) < A - id$ then $\tilde{t} \in (t_k,t_{k+1})$. Moreover $m(\tilde{t}) = A - id$ and $m(t) \leq A$for$t \in [t_k,\tilde{t}\,]$. Let $\bar{t} = \sup\{t \in [t_k,\tilde{t}] | m(t) \leq g(A)\}$. Since $m(\tilde{t}) = \mathrm{A} > g(A) \geq m(t_k)$, then $\bar{t} \in [t_k,\tilde{t}], m(\bar{t}) = g(A)$ and $m(t) \geq g(A)$ for $t \in [\bar{t},\tilde{t}]$. Thus for $t \in [\bar{t},\tilde{t})$ and $\alpha \leq s \leq t$, we have $g(m(s)) \leq g(A) \leq m(t)$. So $g(m(s)) \leq m(t)$ for $t \in [\bar{t},\tilde{t}]$ and $\max\{\alpha, t - q(V(t)) \leq s \leq t\}$. By condition (ii), $m'(t) \leq p(t)c(V(t))$ for $t \in [\bar{t},\tilde{t}]$ and thus (3.4) holds true. However

$$\int_{m(\bar{t})}^{m(\tilde{t})} \frac{ds}{c(s)} = \int_{g(A)}^{A-id} \frac{ds}{c(s)} = \int_{g(A)}^{A} \frac{ds}{c(s)} - \int_{A-id}^{d} \frac{ds}{c(s)} \quad (3.10)$$

Since $\hat{b}(\epsilon) \leq A$ then $g\left(\hat{b}(\epsilon)\right) \leq g(A) \leq A - id < A$ and so $\frac{1}{c(s)} \leq M$for$A - id \leq s \leq A$. Thus from (3.10) we get

$$\int_{m(\bar{t})}^{m(\tilde{t})} \frac{ds}{c(s)} \geq M_2 - \int_{A-id}^{d} M ds = M_2 - dM > M_2 - M2-M1=M1.$$

This contradicts (3.4) and so $(3.9)_i$ holds.

This proves the first part.

The proof of second part is similar.

Suppose that $(3.9)_i$ holds for some $1 \le i \le N$. We prove that

$$m(t) \le A-(i+1)d,\ t \ge t_{l_{i+1}}.\ (3.9)_{i+1}$$

Assume for the sake of contradiction that there exist some $t \ge t_{l_i}$ for which $m(t) > A-(i+1)d$. Then define $\tilde{t} = \inf\{t \ge t_{l_i} \,|\, m(t) > A-(i+1)d\}$ and let $k \ge l_{i+1}$ be chosen so that $\tilde{t} = [t_k, t_{k+1})$. Since $\hat{b}(\epsilon) \le A - id \le \hat{a}(\rho_1)$ then $g(A-id) < (A-id)-d$ and so $m(t_k) \le g(m(t_k^-)) \le g(A-id) < A-(i+1d$. Thus $t \in (tk, tk+1)$.

Moreover $m(\tilde{t}) = A-(i+1)d$ and $m(t) \le A-(i+1)d$ for $t \in [t_k, \tilde{t}]$. Define $\bar{t}$ as before. Since $m(\tilde{t}) = A-(i+1)d > g(A-id) \ge m(t_k)$ then $\bar{t} \in (t_k, \tilde{t})$, $m(\bar{t}) = g(A)$ and $m(t) \ge g(A)$ for $t \in [\bar{t}, \tilde{t}]$. Thus we obtain inequality (3.4) as before. However,

$$\int_{m(\bar{t})}^{m(\tilde{t})} \frac{ds}{c(s)} = \int_{g(A-id)}^{A-(i+1)d} \frac{ds}{c(s)}$$

$$= \int_{g(A-id)}^{A-id} \frac{ds}{c(s)} - \int_{A-(i+1)d}^{A-id} \frac{ds}{c(s)} \qquad (3.11)$$

Since $\hat{b}(\epsilon) \le A-id \le \hat{a}(\rho_1)$ then $g\left(\hat{b}(\epsilon)\right) \le g(A-id) < A-(i+1)d < A-id \le \hat{a}(\rho_1)$ and so $\frac{1}{c(s)} \le M$ for $A-(i+1)d \le s \le A-id$. Thus from (3.11), we get

$$\int_{m(\bar{t})}^{m(\tilde{t})} \frac{ds}{c(s)} \ge M_2 - \int_{A-(i+1)d}^{A-id} M ds = M_2 - dM > M2+M1-M2=M1. \quad (3.12)$$

This contradicts (3.4)and so $(3.9)_{i+1}$ holds.

By the induction, we know that $(3.9)_i$ holds for $i = 1,2,\ldots\ldots, N$. In particular, for $i = N$, we have $\hat{b}(\|x(t)\| \le V(t) = m(t) \le A - Nd \le \hat{a}(\rho_1) - Nd \le \hat{b}(\epsilon), t \ge \sigma + T \ge t_{l_N}$.

Thus

$\|x(t)\| < \epsilon$ for $t \ge \sigma + T$.

This completes the proof.

Theorem 3.2

Assume that there exist functions $a, b, c \in K_1$, $p, q \in PC(R^+, R^+)$, $g, \hat{g} \in K_2$ where $s \le \hat{g}(s) \le g(s)$ for $s > 0$ and $V: [t^*, \infty) \times S(\rho) \to R^+$, where V is continuous on $[t_{k-1}, t_k) \times S(\rho_1)$ for $= 1,2,\ldots.$. Assume that the following conditions hold:

i) $b(\|x\|) \le V(t, x(t)) \le a(\|x\|)$ *for all* $(t,x) \in [t^*, \infty) \times S(\rho)$

ii) $V'(t, x(t)) \le -p(t)c(V(t, x(t))$ *for any solution* $x(t) = x(t, \sigma, \varphi)$ *of (2.1) and (2.2) whenever* $V(t, x(t)) > g(V(s, x(s))$ *for* $\alpha \le s \le t$;

iii) $V(t_k, x + I(t_k, x)) \le \hat{g}(V(t_k^-, x))$ for each $k \in Z^+$ and all $x \in S(\rho_1)$

iv)

$\mu = \inf_{k \in Z^+}\{t_k - t_{k-1}\} > 0$, $M_2 = \sup_{u>0} \int_u^{g(u)} \frac{ds}{c(s)}$ and $M_1 = \inf_{k \in Z^+} \int_{t_k}^{t_k+1} p(s)ds > M_2$

Then the trivial solution of (2.1) and (2.2) is uniformly asymptotically stable.

Example: Consider the equation

$$x'(t) = f(t, x(t)) + g(t, x(t-\tau)) + \int_{-\infty}^{0} h(t, u, x(t+u))du, \quad t \ge 0 \qquad (3.13)$$

$$\Delta x(t_k) = I(t_k, x(t_k^-)), k \in Z^+ \qquad (3.14)$$

where $\tau > 0, f, g \in C(R^+ \times R, R), f(t,0) \equiv 0, |g(t,x)| \le b(t)|x|(b \in C(R^+, R^+))$, $h(t,u,v)$ is continuous on $R^+ \times (-\infty, 0] \times R, |h(t,u,v)| \le m(u)|v|, (m \in C(R^+, R^+))$ and $|x + I(t_k, x)| \le \lambda|x|$ for $x \in R$, where $0 < \lambda < 1$ is a constant. Suppose that there are constants $\mu > 0, L > 0$ such that

$$\left|\frac{f(t,x)}{x}\right| + \lambda^{-1}\left(b(t) + \int_{-\infty}^{0} m(u)du\right) \le L, t \ge 0, \quad x \ne 0 \qquad (3.15)$$

$$t_k - t_{k-1} \le \mu < -\frac{ln\lambda}{L}, k \in Z^+ \qquad (3.16)$$

Then the zero solution of (3.13) and (3.14) is U.A.S

In fact, from (3.15), we can choose a constant $A > 0$ and a continuous function $q: (0,\infty) \to (0,\infty), q$ is non increasing, $q(s) \ge \tau$ such that

$$\int_{-\infty}^{-q(s)} m(u)du \le A\sqrt{s}, t_k - t_{k-1} \le \mu < \frac{-2ln\lambda + A}{2(L+A)}$$

Let $V(t,x) = x^2, g(s) = \lambda^2(s), c(s) = s$. Then

$$V(t_k, x + I(t_k, x)) = [x + I(t_k, x)]^2 \le \lambda^2 x^2 = g(V(t_k^-, x))$$

When $g(V(s, x(s)) < V(t, x(t))$, for $-\infty < s \le t$, we have

$$V'(t, x(.)) \le 2x(t)f(t, x(t)) + 2b(t)|x(t)||x(t-\tau)| + 2|x(t)|\int_{-\infty}^{t} m(v-t)|x(v)|dv$$

$$\le 2|x(t)|^2\left[\left|\frac{f(t, x(t))}{x(t)}\right|\right] + \lambda^{-1}\left(b(t) + \int_{-\infty}^{0} m(u)du\right)$$

$\le 2|x(t)|^2$

And $\int_{g(u)}^{u} \frac{ds}{c(s)} = -2\ln\lambda$, $\int_{t_k}^{t_k+1} 2Lds \le 2L\mu < -2\,ln\,\lambda$.

From Theorem 3.2, we see that the zero solution of (3.13) and (3.14) is uniformly stable. Without any loss of generality, we may assume that $\|x\|^{(-\infty,t]} \le 1$. Thus if

$g(V(s, x(s)) < V(t, x(t))$, for $max\{-\infty, t - q(V(t, x(t)))\} \le s \le t$.

We have

$$V'(t, x(.)) \le 2x(t)f(t, x(t)) + 2b(t)|x(t)||x(t-\tau)| + 2|x(t)|\int_{-\infty}^{t} m(v-t)|x(v)|dv$$

$$\leq 2x(t)f\big(t,x(t)\big) + 2\lambda^{-1}b(t)|x(t)|^2 + 2|x(t)|\int_{t-q(V(t,x(.)))}^{t} m(v-t)|x(v)|dv + 2|x(t)|\int_{-\infty}^{t-q(V(t,x(.)))} m(v-t)|x(v)|dv$$

$$\leq 2|x(t)|^2\left[\left|\frac{f\big(t,x(t)\big)}{x(t)}\right|\right] + \lambda^{-1}\left(b(t)+\int_{-\infty}^{0} m(u)du\right) + 2|x(t)|\int_{-\infty}^{-q(V(t,x(.)))} m(u)du$$

$$\leq\ 2(L+A)|x(t)|^2$$

And $\int_{g(u)}^{u}\frac{ds}{c(s)} = -2\ln\lambda,\ \int_{t_k}^{t_k+1} 2(L+A)ds \leq$ $2(L+A)\tau < -2\ln\lambda$.

From Theorem 3.1, the zero solution of (3.13) and (3.14) is uniformly asymptotically stable.

Conclusion

In the present paper, we conclude that certain impulsive perturbations may make unstable systems uniformly stable, even uniformly asymptotically stable.

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